ISRAEL JOURNAL OF MATHEMATICS, Vol. 43, No. 2, 1982

THE BOUNDARY BEHAVIOR OF FUNCTIONS MEROMORPHIC IN BOUNDED PLANE REGIONS

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ABSTRACT
The object of this note is to extend the classical theorems of Fatou–Nevanlinna, Riesz–Nevanlinna, Riesz–Tsuji, Lusin–Privalov and Lusin–Privalov–Tsuji to bounded plane regions.

The classical theorems of Fatou–Nevanlinna, Riesz–Nevanlinna, Riesz–Tsuji, Lusin–Privalov and Lusin–Privalov–Tsuji are well known (see [1], [6]). In this note these theorems will be extended to bounded plane regions, and those extensions will be given by the Theorem, Corollaries 1, 2, 3 and 4, respectively.

Throughout this note let D be a bounded region in the complex plane, let p be an accessible point on the boundary ∂D of D from D and let L_p be a path in D terminating at p.

Let L and L' be two paths in D terminating at p. In this note we identify the two terminal points of L and L', if for any open disc U centered at p two points $q \in L \cap U$ and $q' \in L' \cap U$ can be joined by an arc included in $D \cap U$, and we distinguish their terminal points, if not. We shall next introduce on D certain regions with the characteristic of angular regions.

Let φ be the projection from the universal covering surface S of D onto D. We map S onto an open unit disc Δ by a univalent holomorphic function ψ. The composition function $g = \varphi \circ \psi^{-1}$ is holomorphic and bounded. By lemma 1 of [3], any lift of a L_p by g^{-1} to Δ is a path terminating at a point $B(L_p)$ on $\partial\Delta$. For any $\varepsilon > 0$, let $R(B(L_p), \varepsilon, r)$ denote the sector region in Δ with radius r, of opening $\pi - \varepsilon$ having vertex at each $B(L_p)$ and bisected by the radius drawn to $B(L_p)$, such that $\overline{R(B(L_p), \varepsilon, r)} \cap \partial\Delta$, where the bar denotes closure, is a single point $B(L_p)$. We denote by $F(B(L_p))$ the family of sector regions $R(B(L_p), \varepsilon, r)$ for each $B(L_p)$, all ε and r. By Lindelöf's theorem, $\overline{g(G)} \cap \partial D$ for any $B(L_p)$ and $G \in F(B(L_p))$ is a single point p.

Received December 10, 1981 and in revised form January 15, 1982

We now give a notation. Suppose as follows: D' is a simply connected plane region bounded by a Jordan curve C'. There is the tangent line to C' at a point p' on C'. L^* is a path in D' terminating at p'. L^* and the tangent line form two angles which have each other distinct openings. Then we denote by $A(L^*)$ the smaller of the openings of the two angles.

Let T be any covering transformation of Δ associated with g such that $g \circ T = g$. T is univalent and holomorphic. For any $B(L_p)$ and $G \in F(B(L_p))$, $\overline{T(G)} \cap \partial\Delta$ is a point or includes an arc. Suppose that $\overline{T(G)} \cap \partial\Delta$ includes an arc. By Riesz' theorem, g reduces to a constant. This contradiction shows that $\overline{T(G)} \cap \partial\Delta$ is a single point. Let s_j $(j = 1, 2)$ denote the two sides, of any $R(B(L_p), \varepsilon, r)$, each having $B(L_p)$ as end point. By Carathéodory's theorem, $A(T(s_j))$ is equal to $A(s_j)$. Next let ψ' be another univalent holomorphic function from S onto another open unit disc Δ'. For any $B(L_p)$ and $G \in F(B(L_p))$, $\overline{(\psi' \circ \psi^{-1})(G)} \cap \partial\Delta'$ is a single point, and $A((\psi' \circ \psi^{-1})(s_j))$ is equal to $A(s_j)$. Therefore the angle of G having vertex at $B(L_p)$ does not depend on the choice of $B(L_p)$ and univalent holomorphic functions from S onto Δ. Henceforth we may thus consider a $B(L_p)$ and a univalent holomorphic function from S onto Δ, when we deal with $F(B(L_p))$.

Suppose that there is a simply connected subregion of D bounded by a Jordan arc C on ∂D and a Jordan arc in D terminating at the two end points of C. Let p lie on C and be distinct from the two end points of C. Suppose that there is the tangent line to C at p. It is then easy to see by choosing a branch of g^{-1} that for any $G \in F(B(L_p))$, $A(g(s_j))$ is equal to $A(s_j)$. Therefore at least in such a special case our regions $g(G)$ have a meaning as angular regions.

Let L and L' be two paths in D terminating at the same point on ∂D. We say that L and L' are homotopic, if there is a continuous family of paths L_s, $0 \leqq s \leqq 1$, in D, each terminating at the same point on ∂D, such that $L_0 = L$ and $L_1 = L'$, or such that $L_0 = L'$ and $L_1 = L$.

For any $G \in F(B(L_p))$, any two paths included in $g(G)$ and terminating at p are homotopic. In the following Remark which is an extension of Lindelöf's theorem, it is seen from the proof of lemma 3 of [3] that L_p and any path included in $g(G)$ and terminating at p are homotopic. By applying Lindelöf's theorem to $f \circ g$, the assertion of the Remark is valid:

Remark. If a function f holomorphic and bounded in D has an asymptotic value c along a L_p, then for any $G \in F(B(L_p))$, $f(z)$ tends uniformly to c as $z \to p$ inside $g(G)$. Here L_p and any path included in $g(G)$ and terminating at p are homotopic.

Let A be an arbitrary set of points on $\partial \Delta$ or ∂D and let $u(K)$ be the harmonic measure of any closed set K, included in A, for Δ or D. If A is a Borel set, then $u(A) = \sup_{K \subset A} u(K)$ is called the harmonic measure of A for Δ or D, and $u(A)$ is also said to be zero or positive, according as $u(A)$ is constant or not (see [2], pp. 6–7, pp. 87–88).

THEOREM. *If f is a meromorphic function with bounded characteristic in D, then for every accessible p on ∂D except possibly for a set of harmonic measure zero and of accessible points on ∂D from D, there exists at least one L_p such that for any $G \in F(B(L_p))$, $f(z)$ tends uniformly to a limit as $z \to p$ inside $g(G)$.*

PROOF. Suppose, on the contrary, that there exists a set A, of positive harmonic measure and of accessible points on ∂D from D, with the property that for any p on A we can find no L_p and $G \in F(B(L_p))$ such that $f(z)$ tends uniformly to a limit as $z \to p$ inside $g(G)$. There exists then a closed set K of positive harmonic measure and included in A.

Let $K_\Delta = \{b \in \partial \Delta \mid g(b) \in K\}$, where $g(b)$ denotes the radial limit of g at a point b on $\partial \Delta$, and let u be the solution of the Dirichlet problem for D with boundary values 1 on K and 0 on $\partial D - K$. Then u is positive and harmonic in D, and $u \circ g$ is also positive and harmonic in Δ. We shall next show that K_Δ is of positive harmonic measure.

By theorem 2.6 of [1], the two exceptional sets E_j $(j = 1, 2)$ of linear measure zero in corollary 1 at page 18 and theorem 2.1 of [1] are Borel sets. If E_j were of positive harmonic measure, then E_j would be of positive linear measure, as the Poisson integral representation shows. Therefore E_j must be of harmonic measure zero. Thus $g(b)$ and $(u \circ g)(b)$ exist at all points b on $\partial \Delta$ except possibly for a set of harmonic measure zero.

Every point on a component, which is a nondegenerate continuum, of D is regular for the Dirichlet problem on D. The set B of irregular points on ∂D is polar and is a Borel set (see [2], theorem 4.7). B is of logarithmic capacity zero (see [2], lemma 5.6). Further, by theorem 1 of [5], $B_\Delta = \{b \in \partial \Delta \mid g(b) \in B\}$ is a Borel set. Therefore it follows from theorem 2.16 of [1] that B_Δ is of harmonic measure zero.

It is now seen that $(u \circ g)(b) = 0$ at almost all b on $\partial \Delta - K_\Delta$ with respect to harmonic measure. Therefore there exists a set E_Δ, of harmonic measure zero and included in Δ, such that $\{b \in \partial \Delta \mid (u \circ g)(b) \neq 0\} \subset K_\Delta \cup E_\Delta$. K_Δ is a Borel set, and if K_Δ were of harmonic measure zero, then by the Poisson integral representation $u \circ g$ would be identically zero in Δ. Thus K_Δ must be of positive harmonic measure.

By the hypothesis of the Theorem there exists a positive harmonic function v such that $\log|f| \leqq v$ in D. Since $f \circ g$ and $v \circ g$ are analytic and harmonic in Δ respectively, $f \circ g$ is a meromorphic function with bounded characteristic in Δ. In theorem 2.18 of [1], radial limits may be replaced by angular limits. It is hence seen that there exists a subset K'_Δ, of positive harmonic measure, of K_Δ with the property that for any point b on K'_Δ, $(f \circ g)(a)$ and $g(a)$ tend uniformly to a limit and a point p on ∂D as $a \to b$ inside any $G \in F(B(L_p))$, where $B(L_p)$ denotes the b. This implies that we can find an accessible p on A, a L_p and a $G \in F(B(L_p))$ such that $f(z)$ tends uniformly to a limit as $z \to p$ inside $g(G)$. This contradiction shows that the assertion of the Theorem is valid.

Corollary 1. *Let A be a set of positive harmonic measure and of accessible points on ∂D from D and let f be a meromorphic function with bounded characteristic in D. If, for all p on A and for all L_p one another being not homotopic, asymptotic values of f along L_p are equal to a constant, then f is identically constant in D.*

Proof. Let K be a closed set of positive harmonic measure and included in A and let K_Δ be the set of terminal points on $\partial\Delta$ of all lifts of all L_p, in the hypothesis of Corollary 1, by g^{-1} to Δ. By Lindelöf's theorem, all the asymptotic values of f along L_p are the radial limits of $f \circ g$ at all points on K_Δ. It is seen from the proof of the Theorem that K_Δ is of positive harmonic measure. By theorem 2.19 of [1], $f \circ g$ reduces to a constant, and hence f is identically constant in D.

It is seen from theorem 2.16 of [1] and the proof of Corollary 1 that the assertion of the following Corollary 2 is valid:

Corollary 2. *Let A be a set of positive harmonic measure and of accessible points on ∂D from D and let f be a function holomorphic and bounded in D. If, for all p on A and for all L_p one another being not homotopic, asymptotic values of f along L_p lie in a set of logarithmic capacity zero, then f is identically constant in D.*

We may apply theorem 2.6 of [1] to functions meromorphic in the open unit disc. Here in the proof of this theorem we use the chordal metric on the Riemann sphere instead of the Euclidean distance on the complex plane. It is easy to see from corollary 1 on page 146 of [1] and the proof of Corollary 1 that the assertion of the following Corollary 3 is valid:

Corollary 3. *Let A be a set of positive harmonic measure and of accessible points on ∂D from D and let f be a function meromorphic in D. If, for all p on A, for* *all L_p one another being not homotopic and for a $G \in F(B(L_p))$, where the opening of the angle of G having vertex at $B(L_p)$ is equal to a positive constant, $f(z)$ tends uniformly to a constant as $z \to p$ inside $g(G)$, then f is identically constant in D.*

It is easy to see from theorem 1 of [6] and the proof of Corollary 1 that the assertion of the following Corollary 4 is valid:

COROLLARY 4. *Let A be a set of positive harmonic measure and of accessible points on ∂D from D and let f be a function meromorphic in D. If, for all p on A, for all L_p one another being not homotopic and for every $G \in F(B(L_p))$, $f(z)$ tends uniformly to a limit, lying in a set of logarithmic capacity zero, as $z \to p$ inside $g(G)$, then f is identically constant in D.*

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